OPTIMUM DISTRIBUTION OF THE WORKING MEDIUM RESISTIVITY TENSOR IN A MAGNETOHYDRODYNAMIC CHANNEL (THE CASE OF NONLOCAL VARIATIONS)

PMM Vol. 35, №3, 1971, pp. 512-531

K. A. LUR'E
(Leningrad)

(Received September 8, 1970)

The plane problem of maximization of the channel external circuit current by the choice of the tensor of the working medium resistivity P is considered on the assumption that tensor P is symmetric and defined by the principal values of $\rho_1(x, y)$ and $\rho_2(x, y)$ corresponding to the principal axes α and β to the angle $\gamma(x, y)$ between the principal and the x-axes. Functions $\rho_i(x, y)$ $(i = 1, 2)$ are subject to the inequalities $0 < \rho_{\min} \leqslant \rho_i(x, y) \leqslant \infty$. Among the necessary conditions for maximum those of Weierstrass and Jacobi, generated by strong local and weak overall variations, respectively, of the control functions (of tensor P), are of particular importance. The Weierstrass condition was analyzed in relation to this problem in [1], where it was shown that $\rho_1(x, y) = \infty$, $\rho_2(x, y) = \rho_{\min}$, can be assumed in optimum operation conditions. Angle $\gamma(x, y)$, generally speaking, is not uniquely defined by the stationarity conditions. The choice of solution is governed by the Jacobi necessary condition which is the subject of the present analysis.

The paper contains three Sections and an Appendix. In Sect. 1 the equations derived in [1] for optimum operation conditions are transformed into a form convenient for analyzing Jacobi's condition. It is shown that under optimum operation conditions the case of arbitrary load R reduces to that of short-circuiting by reducing the external field induction $B(x)$ by a certain constant.

Section 2 contains the derivation of the Jacobi necessary condition.

The case of short-circuiting is considered in Sect. 3. The process of the extremal solution change owing to the violation of Jacobi's condition is illustrated on the example of a channel bounded by horizontal electrodes and vertical insulators. Data defining the gain in the value of the functional, which (gain) is consequent to passing from an extremal solution not satisfying Jacobi's condition to that which satisfies it, are computed for this example.

The possibility of deriving a satisfactory approximation for the optimum value of current by means of horizontal insulating baffles fitted in a channel with a medium of constant scalar resistivity $\rho_{\min}$ is discussed in the conclusion.

Unless otherwise indicated, the notation of [1] is used throughout the following text.

1. Statement of the problem. A plane channel of width 2δ is considered. Its walls are dielectric throughout, except two of its sections of equal length 2λ, located on its opposite sides and facing each other, which are of a perfectly conducting material (Fig. 1). The conducting sections are interconnected across a load R.

A working substance whose resistivity is represented by the varying from point to point symmetric tensor $P_0(x, y)$ moves in the channel at velocity $\mathbf{v}\,(V(y), 0, 0)$. The Cartesian components $\rho_{xx}, \rho_{xy}, \rho_{yy}$ of tensor P_0 are defined by formulas

$$\rho_{xx} = {}^1/_2\,[\rho_1 + \rho_2 + (\rho_1 - \rho_2)\cos 2\gamma]$$
$$\rho_{yy} = {}^1/_2[\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\gamma]$$
$$\rho_{xy} = \rho_{yx} = {}^1/_2\,(\rho_1 - \rho_2)\sin 2\gamma$$

The superposition of the magnetic field $\mathbf{B}\,(0, 0, -B(x))$, $B(x) = B(-x)$ generates in the channel electric current $\mathbf{j}$ (the Cartesian components of this vector are denoted by j_x and j_y) and in the external circuit current I defined by

$$I = \int_{-\lambda}^{\lambda} j_y(x, \pm\delta)\,dx \qquad (1.1)$$

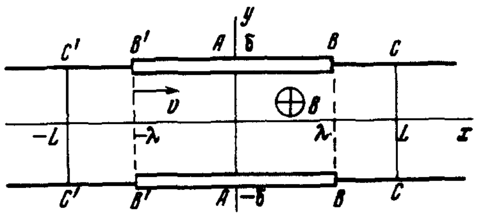

Fig. 1

The equations defining the current distribution in the channel are of the form [1]

$$\operatorname{div}\mathbf{j} = 0, \qquad (1.2)$$

$$P_0\cdot\mathbf{j} = -\operatorname{grad} z^1 + \frac{1}{c}\,\mathbf{v}\times\mathbf{B}$$

Here z^1 denotes the electric field potential.

If the current function is expressed by the relationship $\mathbf{j} = -\operatorname{rot}\mathbf{i}_3 z^2$, then Eqs. (1.2), (1.1) become

$$z_x^1 = \rho_{xx} z_y^2 - \rho_{xy} z_x^2, \quad z_y^1 = \rho_{yx} z_y^2 - \rho_{yy} z_x^2 + c^{-1}VB \qquad (1.3)$$
$$I = z^2(\lambda, \pm\delta) - z^2(-\lambda, \pm\delta)$$

They are supplemented by the boundary conditions (Fig. 1)

$$z^1 = z^1_{\pm} = \text{const along } BB', \qquad z^2 = z^2_- = \text{const along } B'C', C'C'$$
$$z^2 = z^2_+ = \text{const along } BC, CC, \quad z^1_+ - z^1_- = R\,(z^2_+ - z^2_-) \qquad (1.4)$$

If the channel is infinitely long, we must add the condition at infinity

$$z_x^2(\pm\infty, y) = z_y^2(\pm\infty, y) = 0 \qquad (1.5)$$

It is assumed in the following that the channel length is bounded by vertical insulating walls $|x| = L$ permeable to the lengthwise flow of the working substance (Fig. 1).

Let us formulate the problem of maximum stated in the introduction for Eqs. (1.3)–(1.5).

As shown in [1], this problem reduces to the following: determine function $\omega_1(x, y)$ so as to satisfy the equation

$$2h'(\omega_1)\,\Delta\omega_1 + h''(\omega_1)\,(\operatorname{grad}\omega_1)^2 = (c\rho_{\min})^{-1}\,VB_x(x) \qquad (1.6)$$

and the boundary conditions

$$\omega_{1y} = 0 \text{ along } BB', \quad \omega_1 = \omega_{1+} \text{ along } BC, CC, \quad \omega_1 = \omega_{1-} \text{ along } B'C', C'C' \quad (1.7)$$

and the additional condition

$$\rho_{\min}\int_{-\delta}^{\delta}\omega_{1x}(0, y)\,dy = 1 - R\,(\omega_{1+} - \omega_{1-}) \qquad (1.8)$$

which defines the unknown difference $\omega_{1+} - \omega_{1-}$ (here and in the following ω_{1x} and ω_{1y} denote derivatives with respect to x, y, respectively). The implicit form of function $h(\omega_1)$ is given by formula

$$h'(\omega_1) = \frac{1}{1 - R(\omega_{1+} - \omega_{1-})}\{\Psi(\omega_1) - R[h(\omega_{1+}) - h(\omega_{1-})]\} \tag{1.9}$$

$$\Psi(\omega_1) = c^{-1}\int\limits_{L(\omega_1)} VB\,dy$$

The inequality $h'(\omega_1) \geqslant 0$ which is the Weierstrass necessary condition must, also, be satisfied.

The integral in the expression for $\Psi(\omega_1)$ is taken along the line of level $L(\omega_1)$ of function ω_1 which is at the same time the line of current $\mathbf{j}$ owing to the equality

$$z^2 = h(\omega_1) \tag{1.10}$$

The current collected at the electrodes is

$$I = h(\omega_{1+}) - h(\omega_{1-}) \tag{1.11}$$

Problem (1.6)–(1.9) may be conveniently stated in another form. Integrating (1.9) with respect to ω_1, we obtain

$$h'(\omega_1) = \frac{1}{1 - R(\omega_{1+} - \omega_{1-})}\left[\Psi(\omega_1) - R\int\limits_{\omega_{1-}}^{\omega_{1+}} \Psi(\omega_1)\,d\omega_1\right] \tag{1.12}$$

Using the Weierstrass condition $h'(\omega_1) \geqslant 0$, we introduce function $f(\omega_1)$ defined by equation

$$f'(\omega_1) = \sqrt{h'(\omega_1)} \tag{1.13}$$

We consider function $f(\omega_1)$ as the new dependent variable; the relation between f and ω_1 is given by formula (1.13), where in its right-hand side we have to substitute expression (1.12) for $h'(\omega_1)$. Constant values f_+, f_- correspond, respectively, to constant vaues ω_{1+} and ω_{1-} of function ω_1 at the insulators.

Function $B(x)$ is even, while function $h(\omega_1)$ can be assumed odd with respect to x; this also holds for ω_1. Hence, instead of region $CCC'C'$ (Fig. 1), we can consider its right-hand half $CBAABC$ and its boundary AA as an insulator along which $\omega_1 = 0$. Function $f(x, y)$ is considered to be odd with respect to x, and it is assumed that

$$f = 0 \quad \text{along } AA, \qquad f = f_+ \quad \text{along } BCCB \tag{1.14}$$

Taking into consideration the equality

$$\Delta f = \sqrt{h'(\omega_1)}\,\Delta\omega_1 + \frac{h''(\omega_1)}{2\sqrt{h'(\omega_1)}}(\operatorname{grad}\omega_1)^2$$

we represent (1.6) in the form

$$Mf \equiv \Delta f - \frac{1}{2c\rho_{\min} f'(\omega_1)} VB_x(x) = 0 \tag{1.15}$$

Function $f'(\omega_1)$ in the right-hand side of this equation is to be expressed in terms of formulas (1.13) and (1.12), taking into account that ω_1 is odd with respect to x. The boundary condition at the electrodes is of the form $f_y = 0$, and the conditions along the insulators are defined by the equalities (1.14) in which the constant is determined by condition (1.8) which now becomes

$$\rho_{\min}\int\limits_{-\delta}^{\delta} f_x(0, y)\,dy = f'(0)(1 - 2R\omega_{1+}) \tag{1.16}$$

Formula (1.13) and the note following it must be taken into account here.

In the analysis of this problem it can be assumed without loss of generality that $R=0$. In fact, introducing function

$$B_1(x) = B(x) - \frac{2cR}{\int\limits_{L(\omega_1)} V\,dy} \int\limits_0^{\omega_{1+}} \Psi(\omega_1)\,d\omega_1$$

and assuming

$$f = \sqrt{1 - 2R\omega_{1+}}\, f_1$$

we obtain for the determination of $f_1(x, y)$ a boundary value problem which is the same as (1.14)–(1.16), if in the latter we substitute $B_1(x)$ for $B(x)$ and assume $R = 0$.

2. Second variation of Jacobi's necessary condition. Equation (1.6) (or (1.15)) is the Euler equation of the considered problem of optimum. As shown in [1] the optimum tensor P_0 is determined by the principal values $\rho_2 = \rho_{\min}$ and $\rho_1 = \infty$. Angle $\gamma(x, y)$ is defined by formula

$$\operatorname{tg}\gamma = f_y/f_x$$

Under optimum working conditions the processes taking place in the channel are defined by equations [1]

$$0 = z_y^2 \cos\gamma - z_x^2 \sin\gamma \tag{2.1}$$

$$z_y^1 \cos\gamma - z_x^1 \sin\gamma = -\rho_{\min}(z_y^2 \sin\gamma + z_x^2 \cos\gamma) + c^{-1}VB\cos\gamma$$

derived from the input equations (1.2) for $\rho_2 = \rho_{\min}$ and $\rho_1 = \infty$. Equations (2.1) may be conveniently written in the equivalent form [2, 3]

$$z_x^1 = \zeta^1, \qquad z_y^1 = u\zeta^1 - \rho_{\min}(u^2+1)\,\zeta^2 + c^{-1}VB \quad (u = \operatorname{tg}\gamma) \tag{2.2}$$

$$z_x^2 = \zeta^2, \qquad z_y^2 = u\zeta^2$$

The symbols ζ^1 and ζ^2 appearing here must not be confused with those in [1], where they have a different meaning.

It can be shown by conventional methods [1] that the increment δu of the control function relative to its optimum value alters the functional $-I$ by

$$\delta(-I) = -\iint \delta H\,dx\,dy \tag{2.3}$$

Here

$$-\delta H = \frac{\delta u}{\zeta^2}\eta_1[\rho_{\min}(\zeta^2)^2\delta u - \zeta^2\delta\zeta^1 + \zeta^1\delta\zeta^2] + \rho_{\min}(\delta u)^2\delta\zeta^2$$

$$\eta_1 = \omega_{1x}, \qquad \zeta^2 = z_x^2 = h'(\omega_1)\,\omega_{1x} \tag{2.4}$$

and function ω_1 is the solution of problem (1.6)–(1.9), while $\delta\zeta^1$ and $\delta\zeta^2$ denote variations of ζ^1, ζ^2 consequent to the control variation δu. Integration of (2.3) is carried out over the whole basic region.

Formulas (2.3) and (2.4) are exact; unfortunately it is not generally possible to determine the sign of the increment $\delta(-I)$, since for any arbitrary variation δu virtually nothing can be said about $\delta\zeta^1$ and $\delta\zeta^2$. Data on these variations can be obtained by restricting the variation δu by various means which would lead to some necessary condition for a minimum. One of these means namely, the strong variation δu over special sets ("bands") in the xy-plane was investigated in [3]. It yields the Weierstrass' necessary condition for a strong relative minimum.

Another way of restricting δu is to assume this variation to be weak, i.e. small in absolute value, but generally different from zero throughout the basic region. Variations $\delta\zeta^1$ and $\delta\zeta^2$ are of the same order of smallness as δu and satisfy the equations in variations

$$\delta z_x{}^1 = \delta\zeta^1, \quad \delta z_y{}^1 = u\delta\zeta^1 - \rho_{\min}(u^2+1)\,\delta\zeta^2 + (\zeta^1 - 2\rho_{\min} u\zeta^2)\,\delta u$$
$$\delta z_x{}^2 = \delta\zeta^2, \quad \delta z^2{}_y = u\delta\zeta^2 + \zeta^2\delta u \tag{2.5}$$

obtained by varying Eqs. (2.3). The exact expressions (2.3), (2.4), on the other hand, differ for the increment $\delta\,(-I)$ with allowance for relationship (2.2) by a term of higher order from the half of the second variation

$$^1/_2\delta^2(-I) = \iint \frac{\eta_1}{\zeta^2}\,\delta u[\rho_{\min}(\zeta^2)^2\,\delta u - \zeta^2\delta\zeta^1 + \zeta^1\delta\zeta^2]\,dx\,dy \tag{2.6}$$

of this functional. For the functional $-I$ to be minimum its second variation must not be negative, and this is the Jacobi necessary condition. We thus come to the associated problem of the second variation minimum: the find the minimum of functional (2.6) for relationships (2.5) and boundary conditions (Fig. 1)

$$\delta z^1 = \pm\,\delta z_+^1 \text{ along } AB, \quad \delta z^2 = 0 \text{ along } AA, \quad \delta z^2 = \delta z_+^2 \text{ along } BCCB$$
$$\delta z_+^1 = R\delta z_+^2 \tag{2.7}$$

derived by the variation of conditions (1.4).

The Euler equation for this problem is constructed in the conventional manner [2, 3] with the use of Lagrange multipliers $\delta\xi_1$, $\delta\eta_1$, $\delta\xi_2$ and $\delta\eta_2$, corresponding to Eqs. (2.5). We obtain

$$\eta_1\delta u + \delta\xi_1 + u\delta\eta_1 = 0$$
$$-\frac{\eta_1}{\zeta^2}\zeta^1\delta u - \rho_{\min}(u^2+1)\,\delta\eta_1 + \delta\xi_2 + u\delta\eta_2 = 0$$
$$2\eta_1\rho_{\min}\zeta^2\delta u + \frac{\eta_1}{\zeta^2}(\zeta^1\delta\zeta^2 - \zeta^2\delta\zeta^1) - (\zeta^1 - 2\rho_{\min}u\zeta^2)\,\delta\eta_1 - \zeta^2\delta\eta_2 = 0$$
$$\delta\xi_i = -\,\delta\omega_{iy}, \qquad \delta\eta_i = \delta\omega_{ix} \qquad (i = 1, 2) \tag{2.8}$$

If we set

$$\delta z^2 = h_1(\omega_1) + h'(\omega_1)\,\delta\omega_1$$

where function $h'(\omega_1)$ is defined by formula (1.9) and $h_1(\omega_1) = \delta h(\omega_1)$ a variation of function $h(\omega_1)$ is a small function of its argument (the derivative of $h_1'(\omega_1)$ is equal to the variation of the right-hand side of (1.9)), it can be readily shown that Eqs. (2.5) and (2.8) reduce the relationship

$$2\,(h_1' + h''\delta\omega_1)\Delta\omega_1 + 2h'\Delta\delta\omega_1 + 2h''(\nabla\omega_1\cdot\nabla\delta\omega_1) + (h_1'' + h'''\delta\omega_1)\times$$
$$\times(\nabla\omega_1)^2 = 0 \tag{2.9}$$

As should have been expected, this is a variation of (1.6) (Jacobi's equation).

Boundary conditions for (2.9) are immediately obtained from conditions (1.7) and (1.8)

$$\delta\omega_{1y} = 0 \text{ along } AB, \quad \delta\omega_1 = 0 \text{ along } AA, \quad \delta\omega_1 = \delta\omega_{1+} \text{ along } BCCB$$
$$\rho_{\min}\,\delta\int\limits_{L(\omega_1)}\omega_{1n}\,ds = -\,2R\delta\omega_{1+} \tag{2.10}$$

It will be readily seen that the problem defined by (2.9) and (2.10) is equivalent to the boundary value problem obtained by the variation of Eq. (1.5) and related boundary conditions for f.

3. The case of short-circuiting. In this case ($R = 0$) in accordance with (1.9) we have

$$h'(\omega_1) = c^{-1} \int\limits_{L(f)} VB\,dy = \Psi(f) \qquad (3.1)$$

where new parameters are introduced for the lines of current level.

Jacobi's equation can now be written as

$$\Delta\delta f = \frac{VB_x(x)}{2c\rho_{\min}}\,\delta\{[\Psi(f)]^{-1/2}\} \qquad (3.2)$$

The boundary conditions are of the form

$$\delta f_y = 0 \text{ along } AB, \qquad \delta f = 0 \text{ along } AA, \qquad \delta f = \delta f_+ \text{ along } BCCB$$

$$\rho_{\min}\,\delta \int\limits_{L(f)} f_n\,ds = \delta\{[\Psi(f)]^{1/2}\} \qquad (3.3)$$

It can be assumed in the analysis of this problem that (see [1]) throughout the channel $f_n \geqslant 0$, where n is the normal to line $L(f)$ in the direction shown in Fig. 2.

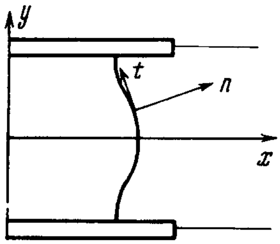

Fig. 2

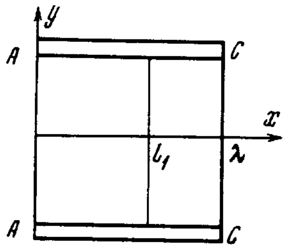

Fig. 3

After variation (see Appendix) (3.2) takes the form

$$\Delta\delta f_P + \frac{VB_x(x_P)}{4c^2\rho_{\min}}[\Psi(f)]^{-3/2} \int\limits_{L(f)} VB_x(x_M)\,\frac{\delta f_P - \delta f_M}{(f_n)_M}\,ds_M = 0 \qquad (3.4)$$

Here δf_P denotes variation δf at point P, and integration is carried out along the line of current $L(f)$ passing through point P (M denotes the point of integration).

The last of conditions (3.3) is of the form (see Appendix)

$$\int\limits_{L(f)} \delta f_n\,ds = 0 \qquad (3.5)$$

This equality can be integrated along any of the lines of current $L(f)$ (to prove this it is sufficient to integrate Eq. (3.4) over the region bounded by two lines of current $L(f)$ connecting the electrodes and by two electrode segments, passing in the integral in the second term of this equation to variables f and s, where f is a parameter of the current line set $L(f)$ and s is the arc length of such line).

Let us analyze problem (3.3)–(3.5) in the particular case in which the horizontal sections BC of insulators (Fig. 1) are absent and the working region is a rectangle bounded by electrodes AC and insulators AA and CC (Fig. 3). If for any $x \in [0, \lambda]$ and $V = \text{const}$, we have $B(x) \geqslant 0$, then function

$$f_0(x) = \frac{1}{\rho_{\min}}\left(\frac{V}{2c\delta}\right)^{1/2}\int_0^x \sqrt{B(x)}\,dx \tag{3.6}$$

satisfies Eq. (1.15) and, also, the conditions $f_y = 0$ at the electrodes and (1.14) at the insulators; condition (1.16) is obviously also satisfied (recall that $R = 0$). The constant f_+ is now

$$f_+ = \frac{1}{\rho_{\min}}\left(\frac{V}{2c\delta}\right)^{1/2}\int_0^\lambda \sqrt{B(x)}\,dx$$

and the lines $f = \text{const}$ are vertical lines connecting the electrodes. The Weierstrass condition $[f_0'(x)]^2 \geqslant 0$ is obviously satisfied.

Jacobi's equation (3.4) becomes

$$\Delta\delta f_P + \frac{[B_x(x)]^2}{4B^2(x)}\left(\delta f_P - \frac{1}{2\delta}\int_{-\delta}^{\delta}\delta f_M\,dy_M\right) = 0 \tag{3.7}$$

and the boundary condition (3.5) is written as

$$\int_{-\delta}^{\delta}(\delta f)_x\,dy = 0 \tag{3.8}$$

The solution for δf_P is sought in the form of series expansion

$$\delta f_P = \sum_{n=0}^{\infty} a_n(x)\cos\frac{n\pi}{\delta}y \tag{3.9}$$

This expression satisfies condition (3.3) at the electrodes. The substitution of (3.9) into (3.7) yields for the coefficients the following equations:

$$\frac{d^2a_n}{dx^2} - \left(\frac{n\pi}{\delta}\right)^2 a_n + \frac{[B_x(x)]^2}{4B^2(x)}a_n = 0 \qquad (n = 1, 2, \ldots) \tag{3.10}$$

$$\frac{d^2a_0}{dx^2} = 0 \tag{3.11}$$

Related boundary conditions

$$a_n(0) = 0 \quad (n = 0, 1, \ldots), \qquad a_0(\lambda) = \delta f_+, \quad a_n(\lambda) = 0 \quad (n = 1, 2, \ldots) \tag{3.12}$$

follow from (3.3) and (3.8). Condition (3.8) yields the equality

$$\frac{da_0(0)}{dx} = 0 \tag{3.13}$$

Only a trivial solution of Eq. (3.11) satisfies the stated conditions, while some of the solutions of Eqs. (3.10) may generally be nontrivial. The existence of such solutions depends on function $B(x)$ and on the index n.

Operators p_n and q acting according to formulas

$$p_n\varphi = \left[-\frac{d^2}{dx^2} + \left(\frac{n\pi}{\delta}\right)^2\right]\varphi \quad (n = 1, 2, \ldots), \quad q\varphi = \frac{[B_x(x)]^2}{4B^2(x)}\varphi$$

are each separately positive-definite along functions φ satisfying conditions (3.12). Let us assume that the positive function $B(x)$ is such that any set of functions with a limited energy norm of operator p_n is compact in the meaning of the energy norm of operator q, i.e.

$$\|\varphi\|_q = \int_0^\lambda \frac{[B_x(x)]^2}{4B^2(x)} \varphi^2\, dx$$

Then Eq. $p_n\varphi - \mu q\varphi = 0$ has an infinite set of eigenvalues [4]

$$0 < \mu_1^{(n)} \leqslant \mu_2^{(n)} \leqslant \ldots$$

And, obviously,

$$\mu_i^{(k)} < \mu_i^{(k+1)}$$

hence $\mu_1^{(1)}$ is the minimum eigenvalue.

If $\mu_1^{(1)} > 1$, the problem defined by (3.10), (3.12) has only trivial solutions for all n, and the related functional (2.6) vanishes. For $\mu_1^{(1)} = 1$ the first of Eqs. (3.10) has a nontrivial solution which satisfies conditions (3.12) and, if $\mu_1^{(1)} < 1$ we can find a function δf such that the related functional (2.8) is negative. In fact, when condition $\mu_1^{(1)} < 1$ is satisfied, there exists a nonzero function $a_{1*}(x)$ which satisfies Eq. (3.10) (for $n = 1$) in the interval $(0,\ l_1)$, where $l_1 < \lambda$, and conditions $a_{1*}(0) = a_{1*}$ and $(l_1) = 0$ at the extremities of that interval. Let us assume

$$a_1(x) = \begin{cases} a_{1*}(x) & (0 \leqslant x \leqslant l_1) \\ 0 & (l_1 \leqslant x \leqslant \lambda) \end{cases}$$

Function

$$\delta f = a_1(x) \cos \frac{\pi y}{\delta} \tag{3.14}$$

satisfies Eq. (3.7) in the interval $(0,\ l_1)$ with respect to x, and the related functional (2.6) vanishes. The latter statement follows from the equality

$$\tfrac{1}{2}\delta^2(-I) = -\oint \delta\omega_{1s}\, \delta z^1\, ds - \oint \delta\omega_{2s} \delta z^2\, ds$$

in which integration is carried out along the boundary of the $AC'C'A$ rectangle (Fig. 3) with formulas (1.10), (1.13), (1.16) and (2.5)–(2.7).

If $\mu_1^{(1)} < 1$, the zero of functional (2.6) is not a minimum. In fact, function (3.14) has a discontinuous first derivative with respect to x; if this is a minimum function, then the conditions of Weierstrass-Erdmann, which in this problem are of the form

$$[\delta\omega_{1s}]_-^+ = 0, \qquad [\delta\omega_{2s}]_-^+ = 0$$

$$\left\{\frac{\eta_1}{\zeta^2}\delta u\, [\rho_{\min}(\zeta^2)^2\, \delta u - \zeta^2\delta\zeta^1 + \zeta^1\delta\zeta^2] + \delta\omega_{1s}\delta z_n^1 + \delta\omega_{2s}\delta z_n^2\right\}_-^+ = 0$$

must be satisfied along the discontinuity line $x = l_1$. The expressions in parentheses vanish to the right of the discontinuity line, while to the left of it they are calculated by formulas

$$\zeta^1 = 0, \qquad \zeta^2 = (c\rho_{\min})^{-1} VB(x), \qquad \eta_1 = (2\rho_{\min}\delta)^{-1}, \qquad u = 0$$

$$\delta u = -\frac{2\rho_{\min}\,\delta}{\sqrt{2c^{-1}VB(x)\,\delta}}\, a_{1*}(x)\frac{\pi}{\delta}\sin\frac{\pi y}{\delta}$$

$$\delta\omega_{2y} = -\frac{\rho_{\min}}{\sqrt{2c^{-1}VB(x)\,\delta}}\left[\frac{da_{1*}(x)}{dx} - \frac{B_x(x)}{2B(x)}\, a_{1*}(x)\right]\cos\frac{\pi y}{\delta}$$

$$\delta\omega_{1y} = -\frac{1}{\sqrt{2c^{-1}VB(x)\,\delta}}\, a_{1*}(x)\,\frac{\pi}{\delta}\sin\frac{\pi y}{\delta}$$

which are readily derived from the foregoing.

The Weierstrass-Erdmann condition now implies $(B(l_1) \neq 0)$

$$\left.\frac{da_{1*}(x)}{dx}\right|_{x=l_1} = 0$$

which cannot be satisfied unless function $a_{1*}(x)$ becomes identically zero in the interval $(0, l_1)$. Hence there must exist a function δf which imparts a negative value to functional (2.6). This proves that for the functional (2.6) to be nonnegative it is necessary that the first eigenvalue of the boundary value problem (3.3)–(3.5) or, what amounts to the same, of the problem (3.10)–(3.12) be not smaller than unity.

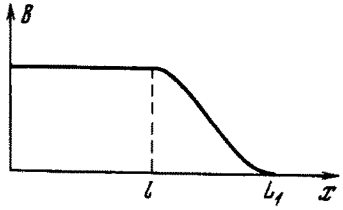

Fig. 4

Note that throughout the region the related δu is not zero, hence δu is not a local variation (as distinct from "variation in a band"). Furthermore, this variation is essentially a two-dimensional one, since Eqs. (3.10) have a nontrivial solution; Eqs. (3.11) has only a trivial solution. In other words, the univariate solution (3.6) satisfies Jacobi's condition with respect to one-dimensional variations but may not satisfy it with respect to two-dimensional variations.

For given λ and δ the value of $\mu_1^{(1)}$ depends on $B(x)$. The case in which $B(x)$ is specified by a curve of the kind shown in Fig. 4 for $l < \lambda$, $L_1 > \lambda$, and in particular for $L_1 = \infty$ is of interest in practical applications. The Weierstrass condition $B(x) \geqslant 0$ is then satisfied, while the fulfilment of Jacobi's condition depends on parameters l, λ, δ, as well as on the pattern of decrease of function $B(x)$ at $x > l$. The latter is satisfied, when $l = 0$ and $B(x)$ decreases linearly; if, however, this function decreases for $x > 0$ according to the law $B_0 e^{-\gamma x}$, where $\gamma > 0$ is a sufficiently great constant ($\gamma > \gamma_0 = 2\pi\lambda^{-1}(1 + \lambda^2\delta^{-2})^{1/2}$), then Jacobi's condition is violated.

In the first of these cases the rate of decrease of function $B(x)$ along section $x > l$ is limited by the Weierstrass condition, that is by the requirement that $B \geqslant 0$ for $x \in [0, \lambda]$. In the second case the Weierstrass condition is always satisfied ($(e^{-\gamma x} > 0$ for any x) and the highest possible γ ($= \gamma_0$) is determined by the Jacobi condition. We have to find optimum solutions for conditions in which function $B(x)$ decreases more rapidly than admitted by the conditions of Weierstrass and Jacobi.

If function $B(x)$ does not increase anywhere and does not change its sign from plus to minus at point x_1 defined by $l < x_1 \leqslant \lambda$, then (3.6) is the optimum solution only on condition that $x_1 = \lambda$. In other words, an insulating wall must be fitted where the nonincreasing function $B(x)$ vanishes.

If, with the Weierstrass condition satisfied, Eq. (3.10) with condition (3.12) has $(1 - \varepsilon)$ $(\varepsilon > 0)$ as its eigenvalue, then Jacobi's condition is not satisfied. In the case of exponential decrease of $B(x)$ described above this occurs when $\gamma > \gamma_0$.

For $\gamma < \gamma_0$ Euler's equation (1.15) has (3.6) as its unique solution satisfying conditions (1.14), (1.16). For $\gamma = \gamma_0$ a "branching" of this solution takes place, while for $\gamma > \gamma_0$ there are generally several solutions of Eq. (1.15). There arises consequently the problem of extending solution (3.6), which is optimum for $\gamma \leqslant \gamma_0$, with respect to parameter γ over $\gamma = \gamma_0$ in such a way that the proposed solution remains optimal, i.e. that it satisfies in any case the necessary conditions of Weierstrass and Jacobi.

Let us find the "small" solutions of this problem, i.e. a local extension of solution (3.6) with respect to parameter γ on the assumption that the remainder $\gamma - \gamma_0$ is fairly small in comparison with γ_0, and that function $B(x)$ is equal to $B_0 e^{-\gamma x}$, for $x > 0$.

Let us set

$$f = f_0 + \varphi, \qquad \gamma = \gamma_0 + k \tag{3.15}$$

and consider the problem (1.14)–(1.16) on the same assumption that $f_0(x)$ is defined by formula (3.6), $V = \text{const}$, $\varphi(x, y)$ is the unknown function, and k / γ_0 is a small positive parameter. In terms of the new variables the problem can be expressed thus:

$$\Phi(\varphi, k) \equiv M(f_0 + \varphi, \gamma_0 + k) = 0 \tag{3.16}$$

$$\varphi = 0 \text{ along } AA, \ \varphi = \varphi_+ \text{ along } CC, \ \varphi_y = 0 \text{ along } AC$$

$$\int_{-\delta}^{\delta} \varphi_x(0, y)\, dy = 0 \tag{3.17}$$

Operator (is the Gateaux derivative of operator $-\Phi$)

$$C = -\Phi_\varphi(0, 0) = -M_\varphi(f_0, \gamma_0)$$

is the Fredholm operator. We write (3.16) in the form

$$C\varphi = R(\varphi, k), \qquad R(\varphi, k) = \Phi(\varphi, k) - \Phi_\varphi(0, 0)\, \varphi \tag{3.18}$$

The homogeneous boundary value problem (3.17) for equation $\Phi_\varphi(0, 0)\, \varphi = 0$ has the nontrivial solution

$$z = a(x) \cos \frac{\pi y}{\delta}$$

where $a(x)$ satisfies Eq. (3.10) and the related conditions (3.12) for $n = 1$. We consider solutions z as normed

$$\int_0^\lambda \int_{-\delta}^{\delta} z^2(x, y)\, dx\, dy = 1$$

For the solvability of the nonhomogeneous boundary value problem $\Phi_\varphi(0, 0)\, \varphi = g(x, y)$ with conditions (3.17) it is necessary and sufficient that

$$\int_0^\lambda \int_{-\delta}^{\delta} gz\, dx\, dy = 0$$

since the related homogeneous problem coincides with its conjugate.

The small solutions of Eq. (3.18) are derived by the Liapunov-Schmidt method described in [5]. Let us write (3.18) as

$$C_1\varphi = F_{01}k + \sum_{i+j \geq 2} F_{ij}\varphi^i k^j + \xi z, \qquad \xi = \int_0^\lambda \int_{-\delta}^{\delta} \varphi z\, dx\, dy \tag{3.19}$$

$$F_{ij} = \frac{1}{i!j!} \frac{\partial^{i+j} R(0, 0)}{\partial \varphi^i \partial k^j}$$

Here F_{ij} is a power operator of order $i + j$ (see [5]) and ξ is a coefficient which remains to be determined. We take into consideration that function $z = a(x) \cos(\pi y / \delta)$ is a simple zero of operator C (see (3.18)). The solution $\varphi(x, y)$ of Eq. (3.19) is sought in the form of series expansion

$$\varphi = \sum_{i=1}^\infty \varphi_{i0} \xi^i + \sum_{i=0}^\infty \xi^i \sum_{j=1}^\infty \varphi_{ij} k^j$$

convergent for sufficiently small $|\xi|$ and k [5]. Substitution of this expansion into

(3.19) yields the relationship

$$\sum_{i=2}^{\infty} L_{i0}\,\xi^i + \sum_{i=0}^{\infty}\xi^i \sum_{j=1}^{\infty} L_{ij}k^j = 0, \qquad L_{ij} = \int_0^{\lambda}\int_{-\delta}^{\delta} \varphi_{ij} z\, dx\, dy \tag{3.20}$$

Functions φ_{ij} are determined from the recurrent system

$$C_1\varphi_{01} = F_{01}, \qquad C_1\varphi_{02} = F_{02} + 2F_{11}\varphi_{01} + F_{20}\varphi_{01}^2 \tag{3.21}$$

$$C_1\varphi_{03} = F_{03} + 3F_{12}\varphi_{01} + 3F_{21}\varphi_{01}^2 + F_{30}\varphi_{01}^3 + 2F_{11}\varphi_{02} + 2F_{20}\varphi_{01}\varphi_{02}$$

$$C_1\varphi_{10} = z, \qquad C_1\varphi_{30} = 2F_{20}\varphi_{10}\varphi_{20} + F_{30}\varphi_{10}^3$$

$$C_1\varphi_{20} = F_{20}\varphi_{10}^2, \qquad C_1\varphi_{11} = 2F_{11}\varphi_{10} + 2F_{20}\varphi_{10}\varphi_{01}$$

$$C_1\varphi_{21} = 2F_{11}\varphi_{20} + 2F_{20}\varphi_{01}\varphi_{02} + 3F_{21}\varphi_{10}^2 + 3F_{30}\varphi_{01}\varphi_{10}^2$$

with boundary conditions (3.17). Substituting the obtained values of these functions into (3.20), we derive formulas defining the constants L_{ij}

$$(u, z) = \int_0^{\lambda}\int_{-\delta}^{\delta} uz\, dx\, dy, \qquad \Gamma = C_1^{-1}$$

We have

$$L_{01} = (F_{01}, \ z), \ \ L_{02} = (F_{02} + 2F_{11}(\Gamma F_{01}) + F_{20}(\Gamma F_{01})^2, \ \ z) \tag{3.22}$$

$$L_{03} = (F_{03} + 3F_{12}(\Gamma F_{01}) + 3F_{21}(\Gamma F_{01})^2 + F_{30}(\Gamma F_{01})^3, \ z) + ([2F_{11} +$$
$$+ 2F_{20}(\Gamma F_{01})]\,\Gamma\,[F_{02} + 2F_{11}(\Gamma F_{01}) + F_{20}(\Gamma F_{01})^2], \ z)$$

$$L_{20} = (F_{20}z^2, \ \ z), \ \ L_{30} = (2F_{20}z\,(\Gamma F_{20}z^2) + F_{30}z^3, \ \ z),$$

$$L_{11} = (2F_{11}z + 2F_{20}z\,(\Gamma F_{01}), \ z)$$

It is shown in the Appendix that formulas

$$F_{01} = \left(\frac{VB_0}{2c\delta}\right)^{1/2} \frac{e^{-1/2\gamma_0 x}}{2\rho_{\min}}\left(1 - \frac{\gamma_0 x}{2}\right), \qquad F_{02} = -\left(\frac{VB_0}{2c\delta}\right)^{1/2} \frac{e^{-1/2\gamma_0 x}}{8\rho_{\min}}\, x\left(1 + \frac{\gamma_0 x}{2}\right)$$

$$F_{11}\varphi k = \frac{\gamma_0 k}{2}\left[\varphi - \frac{1}{2\delta}\int_{-\delta}^{\delta} \varphi_M(x, y_M)\, dy_M\right]$$

$$F_{20}uv = \frac{\gamma_0^2\rho_{\min}}{8}\left(\frac{c}{2VB_0\delta}\right)^{1/2} e^{1/2\gamma_0 x}\left\{\int_{-\delta}^{\delta} [(u-u_M)(v_M)_x + (v - v_M)(u_M)_x]\, dy_M + \right.$$
$$\left. + \frac{\gamma_0}{4}\int_{-\delta}^{\delta}(u-u_M)(v-v_M)\, dy_M - \frac{3}{8}\frac{\gamma_0}{\delta}\int_{-\delta}^{\delta}(u-u_M)\, dy_M \int_{-\delta}^{\delta}(v - v_M)\, dy_M\right\}$$

$$\tag{3.23}$$

$$\frac{4VB_0}{\gamma_0^2\rho_{\min}^2 c}\, e^{-\gamma_0 x} F_{30}u^3 = \int_{-\delta}^{\delta}(u-u_M)(u_M)_x^2\, dy_M - \frac{1}{2}\int_{-\delta}^{\delta}(u-u_M)^2 (u_M)_{xx}\, dy_M +$$
$$+ \frac{\gamma_0}{4}\int_{-\delta}^{\delta}(u-u_M)^2(u_M)_x\, dy_M - \frac{3}{4}\frac{\gamma_0}{\delta}\int_{-\delta}^{\delta}(u-u_M)\, dy_M \int_{-\delta}^{\delta}(u-u_M)(u_M)_x dy_M -$$
$$- \frac{3}{16}\frac{\gamma_0^2}{\delta}\int_{-\delta}^{\delta}(u-u_M)\, dy_M \int_{-\delta}^{\delta}(u-u_M)^2\, dy_M + \frac{5}{32}\frac{\gamma_0^2}{\delta^2}\left[\int_{-\delta}^{\delta}(u-u_M)\, dy_M\right]^3$$

are valid.

These relationships together with (3.22) show that $L_{0i} = 0$ $(i = 1, 2, \ldots)$; it can also be checked that $L_{20} = 0$. Hence the branching equation (3.20) results in the following controlling equation [5]:

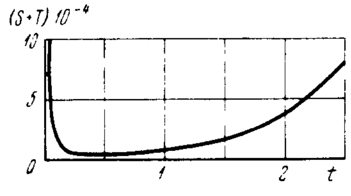

Fig. 5

$$L_{30}\xi^2 + L_{11}k = 0$$

Both roots $\xi_{1,2} = \pm(-kL_{11} / L_{30})^{1/2}$ of this equation are real, since it can be shown (we omit the proof owing to the limited space) that

$$\xi_{1,2} = \pm\frac{2\lambda^2}{\pi}\left\{\frac{{}^1/_2\pi\sqrt{1+t}+2}{(1+t)[S(t)+T(t)]}\frac{VB_0}{c\rho_{\min}^2}k\right\}^{1/2}, \quad t = \frac{\lambda^2}{\delta^2}$$

$$S(t) = \frac{1}{\pi^2(2+t)(5+t)}\left\{\left[\frac{35}{8}\pi^2(1+t) - 1 + t + \frac{21}{64}(1-t^2) + \right.\right.$$
$$\left. + \frac{5}{4}\pi^2(5+t)^2 + 8\pi^2\frac{5+t}{1+t}(2+t)\right](e^{2\pi\sqrt{1+t}} - 1) -$$
$$\left. - 72\pi^2(2+t)(e^{\pi\sqrt{1+t}} - 1)\right\}$$

$$T(t) = \frac{1+t}{16(1-t)}\left\{-\frac{e^{2\pi\sqrt{1+t}}-1}{2+t} + \left(\frac{1}{5+t} + \frac{1}{1-t}\right)\frac{2}{5+t}\times\right.$$
$$\times\left(5t + 1 - 3\frac{e^{\pi\sqrt{1+t}} - \cos\pi\sqrt{1-3t}}{\sin\pi\sqrt{1-3t}}\sqrt{(1+t)(1-3t)}\right)(e^{\pi\sqrt{1+t}}\times$$
$$\times\cos\pi\sqrt{1-3t} - 1) +$$
$$+\left[3\sqrt{(1+t)(1-3t)} + (5t+1)\frac{e^{\pi\sqrt{1+t}} - \cos\pi\sqrt{1-3t}}{\sin\pi\sqrt{1-3t}}\right]\times$$
$$\left.\times e^{\pi\sqrt{1+t}}\sin\pi\sqrt{1-3t}\right\} \qquad (3.24)$$

It will be seen from the curve of function $S(t) + T(t)$ that $S(t) + T(t) > 0$ for all $t \geqslant 0$.

The small solutions of Eq. (1.15) are of the form

$$f_{1,2} = f_0 + \xi_{1,2}\left(\frac{2}{\lambda\delta}\right)^{1/2}\sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\delta} + O(k) =$$
$$= \frac{2}{\gamma\rho_{\min}}\left(\frac{VB_0}{2c\delta}\right)^{1/2}(1 - e^{-1/2\gamma x}) \pm \frac{4\lambda}{\pi\rho_{\min}}\left\{\frac{VB_0}{2c\delta}\frac{{}^1/_2\pi\sqrt{1+t}+2}{(1+t)[S(t)+T(t)]}k\lambda\right\}^{1/2}\times$$
$$\times\sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\delta} + O(k) = f_0 \pm \omega k^{1/2} + O(k) \qquad (3.25)$$

The direct verification of Jacobi's condition is difficult; it is much easier to calculate the total current increment consequent on the transition to the small solution (3.25). The current corresponding to the invariate solution (3.6) is

$$I = \frac{2VB_0}{\gamma c\rho_{\min}}(1 - e^{-\gamma\lambda}) \qquad (3.26)$$

The j_y-component of vector $\mathbf{j}$ is defined (see (1.10) and (1.13)) by formula

$$j_y = [\Psi(f)]^{1/2} f_x, \qquad \Psi(f) = c^{-1} V \int\limits_{L(f)} B\, dy$$

The following relationships

$$[\Psi(f)]^{1/2} = [\Psi(f_0)]^{1/2} + \frac{V}{2c}[\Psi(f_0)]^{-1/2}\Big\{\pm \rho_{\min}\Big(\frac{2kc\delta}{VB_0}\Big)^{1/2}\int\limits_{-\delta}^{\delta} B_x(x)\, e^{1/2\gamma_0 x}(\omega -$$

$$-\,\omega_M)\, dy_M + \rho_{\min} k\Big(\frac{2c\delta}{VB_0}\Big)^{1/2}\int\limits_{-\delta}^{\delta} B_x(x)\, e^{1/2\gamma_0 x}(\chi - \chi_M)\, dy_M -$$

$$-\,\rho_{\min}^2 \frac{2c\delta}{VB_0} k \int\limits_{-\delta}^{\delta} B_x(x)\, e^{\gamma_0 x}(\omega - \omega_M)(\omega_M)_x\, dy_M +$$

$$+\,\rho_{\min}^2 \frac{2c\delta}{VB_0}\frac{\gamma_0 k}{4}\int\limits_{-\delta}^{\delta} B_x(x)\, e^{\gamma_0 x}(\omega - \omega_M)^2\, dy_M +$$

$$+\,\frac{1}{2}\rho_{\min}^2 \frac{2c\delta}{VB_0} k \int\limits_{-\delta}^{\delta} B_{xx}(x)\, e^{\gamma_0 x}(\omega - \omega_M)^2\, dy_M\Big\} -$$

$$-\,\frac{\rho_{\min}^2 V^2}{8c^2}\frac{2c\delta}{VB_0} k [\Psi(f_0)]^{-3/2}\Big[\int\limits_{-\delta}^{\delta} B_x(x)\, e^{1/2\gamma_0 x}(\omega - \omega_M)\, dy_M\Big]^2 + o(k) =$$

$$= [\Psi(f_0)]^{1/2} \pm pk^{1/2} + qk + o(k)$$

$$B = B_0 e^{-\gamma_0 x} \tag{3.27}$$

are valid (see Appendix). The increment k of exponent γ has been omitted in the last formula; this can be done by calculating the total current increment due to the two-dimensional character of solution (3.25).

We write the expression for δf in the form

$$\delta f = \pm \omega k^{1/2} + \chi k, \qquad \chi = b \sin\frac{\pi x}{\lambda}\cos\frac{\pi y}{\delta}$$

Here the first term in the right-hand side is defined by formula (3.25); the coefficient b, although unknown, has to be retained for the sake of calculation accuracy; it will be shown subsequently that it is not necessary to determine it explicitly.

With the use of formula $f_x = (f_0)_x \pm \omega_x k^{1/2} + \chi_x k$ we find the increment of the j_y-component of vector $\mathbf{j}$

$$\delta j_y = [\Psi(f)]^{1/2} f_x - [\Psi(f_0)](f_0)_x = \pm\{[\Psi(f_0)]^{1/2}\omega_x + p(f_{00})_x\} k^{1/2} +$$
$$+ \{[\Psi(f_0)]^{1/2}\chi_x + p\omega_x + q(f_{00})_x\} k + o(k)$$

$$f_{00} = \frac{2}{\gamma_0 \rho_{\min}}\Big(\frac{VB_0}{2c\delta}\Big)^{1/2}(1 - e^{-1/2\gamma_0 x}) \tag{3.28}$$

We have

$$p = -\frac{\gamma_0 \rho_{\min} V}{2c}[\Psi(f_0)]^{-1/2}\Big(\frac{2c\delta}{VB_0}\Big)^{1/2} B(x)\, e^{1/2\gamma_0 x}\int\limits_{-\delta}^{\delta}(\omega - \omega_M)\, dy_M = -\gamma_0 \rho_{\min}\omega\delta$$

From this follows the expression for the coefficient at $\pm k^{1/2}$ in (3.28)

$$\Big(\frac{2VB_0\delta}{c}\Big)^{1/2} e^{-1/2\gamma_0 x}\Big(\omega_x - \frac{\gamma_0}{2}\omega\Big) \tag{3.29}$$

Integration of this expression (for $y = \delta$) with respect to x from zero to λ yields zero; hence the total current increment is determined by the term in (3.28) linear with respect to k.

We separate in the expression for $q(f_{00})_x$ (see (3.27) and (3.28)) the term

$$q_1 = \frac{1}{2}\left(\frac{V}{2B_0c\delta}\right)^{1/2} \int_{-\delta}^{\delta} B_x(x)\, e^{1/2\gamma_0 x} (\chi - \chi_M)\, dy_M$$

which, when combined with $[\Psi(f_0)]^{1/2}\chi_x$ (see (3.28)), yields an expression which differs from (3.29) by a constant factor, hence the related contribution to the total current is zero. It remains to calculate the integral

$$2k \int_0^\lambda (q - q_1)\,|_{y=\delta}\, (f_{00})_x\, dx$$

We have

$$q - q_1 = \frac{8\lambda^2}{\pi} \frac{1/2\,\pi\sqrt{1+t}+2}{\sqrt{1+t}\,[S(t)+T(t)]} \left(\frac{VB_0\delta}{2c}\right)^{1/2} e^{1/2\gamma_0 x}\left(\frac{\gamma_0}{4} - \frac{\gamma_0}{4}\cos\frac{2\pi x}{\lambda} - \frac{\pi}{\lambda}\sin\frac{2\pi x}{\lambda}\right)$$

$$2k \int_0^\lambda (q - q_1)\,|_{y=\delta}\, (f_{00})_x\, dx = 4\lambda^2 k \frac{VB_0}{c\rho_{\min}} \frac{1/2\,\pi\sqrt{1+t}+2}{S(t)+T(t)}$$

The increment of total current is thus

$$\delta I = 4\lambda^2 k \frac{VB_0}{c\rho_{\min}} \frac{1/2\,\pi\sqrt{1+t}+2}{S(t)+T(t)}$$

Function $\Delta(t) = (\delta I / Ik\lambda)10^2$ is shown in Fig. 6 in the form of a curve. It will be seen from this curve that for $k\lambda = 2 < 2\pi < \gamma_0\lambda$ the transition to the small solution is related to an up to 5% increase of current flow from the electrodes.

It appears that the current increase is in the main independent of the choice of the small solution (of the choice of sign in formula (3.25)). The difference between these solutions can manifest itself in terms of higher order with respect to k in the current density expression.

The physical reason of current increase shown by the two-dimensional solution is that in a fairly rapidly decreasing field $B(x)$ it can be advantageous to utilize to a lesser extent the region of small values of $B(x)$, simultaneously reducing in the channel the effective resistance to current generated in its middle part, where the field is sufficiently strong. Such conditions of current flow are established by solution (3.25) whose lines of levels (lines of current) corresponding to the upper and lower signs in (3.25) are shown in Figs. 7a and b, respectively.

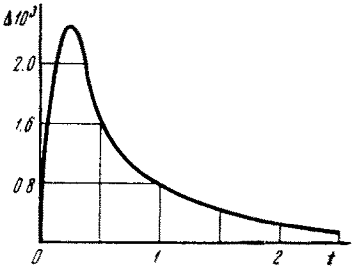

Fig. 6

An approximation to the resistivity tensor distribution defined by the equalities $\rho_1 = \infty$ and $\rho_2 = \rho_{\min}$ can be obtained by introducing in the stream of an isotropically conducting medium of resistivity $\rho = \rho_{\min}$ a set of fairly closely spaced thin insulating baffles permeable to the fluid. The optimum shape and the arrangement of such baffles in the stream of working fluid are dictated by the configuration of the lines of current in the optimum mode of tensor resistivity. If the channel geometric parameters and the functions $V(y)$ and $B(x)$ are specified, the optimum mode can be derived by methods described in [1] and in this paper.

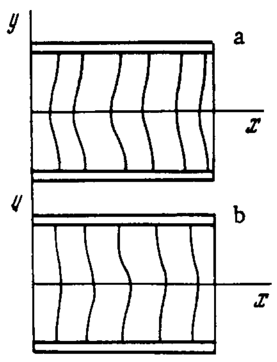

Fig. 7

Other known solutions [6, 7] indicate the feasibility of increasing the current by fitting the channel with horizontal nonconducting baffles parallel to the channel axis (conventional analytical methods can be used for solving the problem of current distribution only for this arrangement of baffles). The horizontal arrangement of baffles is not the best. The question how close is it possible to come to the optimum current output is resolved with the use of methods described here.

Let us assume that function $B(x)$ is specified by a curve of the kind shown in Fig. 4 and consider the mode of homogeneous isotropic resistivity $\rho = \rho_{\min} = \mathrm{const}$ as the nonoptimal input mode 1. The channel is assumed to be bounded at its ends by the nonconducting walls CC and $C'C'$ (Fig. 1). We denote by I_1 and I_2, respectively, the current flowing under conditions of mode 1 and that under those of the optimum tensor mode 2, in otherwise equal conditions. The rest $I_2 - I_1$ is, generally speaking, due to the violation in mode 1 of the Weierstrass and Jacobi necessary conditions. The optimizing effect of baffles is determined by the extent to which these restore the violated necessary conditions.

The Weierstrass condition is of local character, and can be restored with a certain degree of precision by changing locally the resistivity by fitting horizontal baffles in the zone of field decrease. These baffles inhibit the formation of current eddies in such zones. The optimum current distribution (mode 2) is entirely free of eddies [1]. Vector lines j in the case of a channel with baffles are the closer to lines j in mode 2 the greater the ratio of the length of the decreasing field zone to the channel width. When this ratio is high, a satisfactory adherence to the Weierstrass condition can be expected in the case of a channel with baffles, while for a channel with short zones of a field decrease this condition is satisfied to a lesser extent.

If the introduction of baffles results in a current distribution which reasonably accurately satisfies not only the Weierstrass but, also, the Jacobi condition, it can be considered that a satisfactory approximation of current output to I_2 has been achieved.

However in a number of cases the Weierstrass condition is already satisfied in the input mode 1, while that of Jacobi is not. In such cases local resistivity changes (including the introduction of horizontal baffles) will not satisfy the latter condition even approximately.

This can be illustrated by the following example. If the channel is bounded by horizontal electrodes and vertical insulators, then for $B(x) \geqslant 0$ solution (3.6) satisfies the Weierstrass condition throughout that channel independently of its length. If at the same time $B = B_0 e^{-\gamma x}$, then for $\gamma\lambda > 2\pi (1 + \lambda^2\delta^{-2})^{1/2}$ solution (3.6) does not satisfy the Jacobi condition, while solution (3.25), the transition to which shows an increased current output, satisfies the latter.

On the other hand, the introduction into the channel filled with a uniformly conducting medium (Fig. 3) of one or more nonconducting baffles (horizontal or otherwise), or the lowering of the electrical conductance in any part of the channel, will only decrease the current output at the electrodes.

The increased current output shown by Solutions (3.25) is due to the effect of the total tensor resistivity zone. The substitution of a sequence of baffles (whose shape is determined by Figs. 7a and b) for this zone increases the current owing to the interaction of charges emanating from the baffles.

Appendix. Calculation of the first coefficients of expansion (3.19). Prior to variation a line of current $L(f)$ passes through point P of the basic region; thereafter the line of current $L_1(F)$ passes though the same point (Fig. 8). These lines are assumed to be close to each other in space C_1.

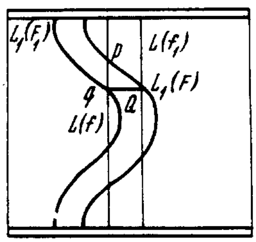

Fig. 8

We define the position of an arbitrary point Q in the neighborhood of line $L(f)$ by coordinates s, ζ, with s being the length of arc of line $L(f)$ between points N and q which is the base of the normal to line $L(f)$ passing through point Q, and ζ being the projection of vector qQ onto the normal $\mathbf{n}$ to the line $L(f)$ at point q. The directions of unit vectors $\mathbf{t}$ and $\mathbf{n}$, respectively, the tangent and the normal to line $L(f)$ are defined by formulas

$$\mathbf{t} = (x_s, y_s), \qquad \mathbf{n} = (y_s, -x_s)$$

The radius vector $\mathbf{R}$ of point Q is related to the radius vector $\mathbf{r}$ of point q by the expression

$$\mathbf{R} = \mathbf{r} + \mathbf{n}\zeta$$

Using the Frenet formula

$$\frac{d\mathbf{n}}{ds} = \frac{\mathbf{t}}{\rho}$$

where ρ denotes the radius of curvature of line $L(f)$ at point q, we obtain the relationship between the differentials

$$d\mathbf{R} = \mathbf{t}\left(1 + \frac{\zeta}{\rho}\right) ds + \mathbf{n}\, d\zeta$$

From this we obtain the following expressions for the Lamé coefficients:

$$h_s = 1 + \zeta / \rho, \qquad h_\zeta = 1$$

If point Q moves along the line $L_1(F)$ ($F = \text{const}$), then along that line

$$d\zeta = -(F_s / F_\zeta)\, ds$$

Simultaneously we obtain

$$d\mathbf{R} = [(1 + \zeta/\rho)\,\mathbf{t} - (F_s / F_\zeta)\,\mathbf{n}]\, ds = \mathbf{T} dS$$

Here $\mathbf{T}$ denotes the unit vector of the tangent to $L_1(F)$ and dS the differential of an arc of that curve. The curve $L(f_1)$ of the basic (old) set (Fig. 8) passes through point Q; variations

$$\varphi_P = F - f, \qquad \varphi_Q = F - f_1$$

are at points P and Q related by

$$\varphi_P = \varphi_Q + f_\zeta \zeta + {}^1/_2 f_{\zeta\zeta}\zeta^2 + {}^1/_6 f_{\zeta\zeta\zeta}\zeta^3 + o(\zeta^3) \qquad \text{(A.1)}$$

and the derivatives f_ζ, $f_{\zeta\zeta}$ and $f_{\zeta\zeta\zeta}$ in the right-hand side of the last equalities are calculated at point q (Fig. 8). Now, assuming that ζ, φ_P, φ_Q, φ_q, $(\varphi_q)_\zeta$ and $(\varphi_q)_{\zeta\zeta}$ are of the same order of magnitude, we write the series expansion

$$\varphi_Q = \varphi_q + (\varphi_q)_\zeta \zeta + {}^1/_2 (\varphi_q)_{\zeta\zeta}\zeta^2 + o(\zeta^3)$$

The substitution of this expansion into (A.1) yields the equation in ζ (for brevity we shall write forthwith φ for φ_Q)

$$\varphi_P - \varphi = (f_\zeta + \varphi_\zeta)\zeta + {}^1/_2 (f_{\zeta\zeta} + \varphi_{\zeta\zeta})\zeta^2 + {}^1/_6 f_{\zeta\zeta\zeta}\zeta^3 + o(\zeta^3) \qquad (A.2)$$

Let us find the root of this equation which would differ only slightly from $(\varphi_P - \varphi)/f_\zeta$. For this we set $\zeta = (\varphi_P - \varphi)/f_\zeta + \varepsilon$, where $\varepsilon = o(\varphi)$. Substituting this into (A.2) and neglecting terms of order higher than three, we obtain the equality

$$0 = \varepsilon\left(f_\zeta + \varphi_\zeta + f_{\zeta\zeta}\frac{\varphi_P - \varphi}{f_\zeta}\right) + \frac{\varphi_P - \varphi}{f_\zeta}\left\{\varphi_\zeta + \frac{1}{2}\,\frac{\varphi_P - \varphi}{f_\zeta}\left[f_{\zeta\zeta} + \varphi_{\zeta\zeta} + \frac{1}{3}\,\frac{f_{\zeta\zeta\zeta}}{f_\zeta}(\varphi_P - \varphi)\right]\right\}$$

Solving this equation for ε, we obtain

$$\zeta = \frac{\varphi_P - \varphi}{f_\zeta} - \frac{\varphi_P - \varphi}{f_\zeta^2}\left(\varphi_\zeta + \frac{1}{2} f_{\zeta\zeta}\frac{\varphi_P - \varphi}{f_\zeta}\right) - \frac{\varphi_P - \varphi}{f_\zeta^3}\left[-\varphi_\zeta^2 + \frac{1}{2}\varphi_{\zeta\zeta}(\varphi_P - \varphi) - \right.$$
$$\left. - \frac{3}{2}\,\frac{f_{\zeta\zeta}}{f_\zeta}\varphi_\zeta(\varphi_P - \varphi) + \frac{1}{6}\,\frac{f_{\zeta\zeta\zeta}}{f_\zeta}(\varphi_P - \varphi)^2 - \frac{1}{2}\,\frac{f_{\zeta\zeta}^2}{f_\zeta^2}(\varphi_P - \varphi)^2\right] \qquad (A.3)$$

Operator $\Phi(\varphi, k)$ (see (3.16)) is defined by the equality ($V = \text{const}$)

$$\Phi(\varphi, k) \equiv \Delta F - \frac{VB_x(x)}{2c\rho_{\min}}[\Psi(F)]^{-1/2} = \Delta f - \frac{VB_x(x)}{2c\rho_{\min}}[\Psi(f)]^{-1/2} +$$
$$+ \Delta\varphi_P - \frac{VB_x(x)}{4c\rho_{\min}}\left\{-[\Psi(f)]^{-3/2}\delta\Psi + \frac{3}{4}[\Psi(f)]^{-5/2}(\delta\Psi)^2 - \right.$$
$$\left. - \frac{5}{8}[\Psi(f)]^{-7/2}(\delta\Psi)^3 + o[(\delta\Psi)^3]\right\} \qquad (A.4)$$

where in the calculation of variation

$$\delta\Psi = \frac{V}{c}\int\limits_{L_1(F)} B\,dy - \frac{V}{c}\int\limits_{L(f)} B\,dy$$

only terms of up to and including the third order with respect to ζ are to be retained.

Along the $L_1(F)$ line $\quad dy\,|_{L_1(F)} = (1 + \zeta/\rho)\,dy - \zeta^{\cdot} dx$

Here dx and dy are projections of $\mathbf{t}ds$ onto the x- and y-axes, respectively, ζ is the coordinate of a point of curve $L_1(F)$, and $\zeta^{\cdot} = d\zeta/ds = -F_s/F_\zeta$; the differentials dx and dy are related by the equation of curve $L(f)$: $f_x dx + f_y dy = 0$.

In the following we assume that $f = f(x)$, which implies that the set consists of vertical straight lines. In this case along the curve $L_1(F)$ $dy\,|_{L_1(F)} = dy$, hence

$$\int\limits_{L_1(F)} B\,dy - \int\limits_{L(F)} B\,dy = \int\limits_{-\delta}^{\delta}\left(B_x\zeta + \frac{1}{2}B_{xx}\zeta^2 + \frac{1}{6}B_{xxx}\zeta^3 + \ldots\right)dy$$

For the operator $\Phi(\varphi, k)$, after simple transformations, we obtain the expression

$$\Phi(\varphi, k) = \Delta\varphi_P - \frac{B_x'(x)}{8\rho_{\min}B\delta}\left(\frac{V}{2Bc\delta}\right)^{1/2}\left[B_x(x)\int\limits_{-\delta}^{\delta}\zeta\,dy + \frac{1}{2}B_{xx}\int\limits_{-\delta}^{\delta}\zeta^2 dy + \right.$$
$$+ \frac{1}{6}B_{xxx}\int\limits_{-\delta}^{\delta}\zeta^3 dy - \frac{3}{8}\,\frac{B_x^2(x)}{B\delta}\left(\int\limits_{-\delta}^{\delta}\zeta\,dy\right)^2 - \frac{3}{8}\,\frac{B_x(x)B_{xx}(x)}{B\delta} \times$$

$$\times\left(\int_{-\delta}^{\delta}\zeta dy\right)\left(\int_{-\delta}^{\delta}\zeta dy\right)+\frac{5}{32}\frac{B_x^3(x)}{B^2\delta^2}\left(\int_{-\delta}^{\delta}\zeta dy\right)^3\Big]+o(\zeta^3)$$

For the function $f=f_0(x)$ defined by formula (3.6) we construct the expression for (A.3) in the form

$$\zeta=\rho_{\min}\left(\frac{2c\delta}{VB_0}\right)^{1/2}e^{1/2\gamma x}\Big\{(\varphi_P-\varphi)-\rho_{\min}\left(\frac{2c\delta}{VB_0}\right)^{1/2}e^{1/2\gamma x}(\varphi_P-\varphi)\Big[\varphi_\zeta-\frac{\gamma}{4}(\varphi_P-\varphi)\Big]-$$
$$-\rho_{\min}^2\frac{2c\delta}{VB_0}e^{\gamma x}(\varphi_P-\varphi)\Big[-\varphi_\zeta^2+\frac{1}{2}\varphi_{\zeta\zeta}(\varphi_P-\varphi)+\frac{3}{4}\gamma\varphi_\zeta(\varphi_P-\varphi)-\frac{\gamma^2}{12}(\varphi_P-\varphi)^2\Big]\Big\}$$

Substituting this into (A.4), we obtain formula

$$-\Delta\varphi_P-\frac{\gamma_0^2}{4}\left(\varphi_P-\frac{1}{2\delta}\int_{-\delta}^{\delta}\varphi dy\right)=\frac{k}{4}(2\gamma_0+k)\left(\varphi_P-\frac{1}{2\delta}\int_{-\delta}^{\delta}\varphi dy\right)-\frac{\gamma_0^2\rho_{\min}}{8\delta}\left(\frac{2c\delta}{VB_0}\right)^{1/2}\times$$
$$\times e^{1/2\gamma_0 x}\left\{\int_{-\delta}^{\delta}\varphi_\zeta(\varphi_P-\varphi)\,dy+\frac{\gamma_0}{4}\int_{-\delta}^{\delta}(\varphi_P-\varphi)^2dy-\frac{3}{8}\frac{\gamma_0}{\delta}\Big[\int_{-\delta}^{\delta}(\varphi_P-\varphi)\,dy\Big]^2\right\}-$$
$$-\frac{\gamma_0^2\rho_{\min}^2c}{4VB_0}e^{\gamma_0 x}\Big\{-\int_{-\delta}^{\delta}\varphi_\zeta^2(\varphi_P-\varphi)\,dy+\frac{1}{2}\int_{-\delta}^{\delta}\varphi_{\zeta\zeta}(\varphi_P-\varphi)^2dy-$$
$$-\frac{\gamma_0}{4}\int_{-\delta}^{\delta}\varphi_\zeta(\varphi_P-\varphi)^2dy+\frac{3}{4}\frac{\gamma_0}{\delta}\int_{-\delta}^{\delta}(\varphi_P-\varphi)\,dy\int_{-\delta}^{\delta}(\varphi_P-\varphi)\,\varphi_\zeta\,dy+$$
$$+\frac{3}{16}\frac{\gamma_0^2}{\delta}\int_{-\delta}^{\delta}(\varphi_P-\varphi)\,dy\int_{-\delta}^{\delta}(\varphi_P-\varphi)^2dy-\frac{5}{8}\frac{\gamma_0^2}{4\delta^2}\Big[\int_{-\delta}^{\delta}(\varphi_P-\varphi)\,dy\Big]^3\Big\}+\ldots+$$
$$+\frac{1}{2\rho_{\min}}\left(\frac{VB_0}{2c\delta}\right)^{1/2}e^{-1/2\gamma_0 x}\left(1-\frac{\gamma_0 x}{2}\right)k-\frac{1}{8\rho_{\min}}\left(\frac{VB_0}{2c\delta}\right)^{1/2}xe^{-1/2\gamma_0 x}\left(1+\frac{\gamma_0 x}{2}\right)k^2$$

From this follow the equalities (3.23) defining F_{ij}.

The determination of L_{ij} (see (3.20)) involves the use of formula (3.22). Since this requires very cumbersome calculations, it is omitted here.

BIBLIOGRAPHY

1. Lur'e, K. A., On the optimal distribution of the resistivity tensor of the working substance in a magnetohydrodynamic channel. PMM, Vol. 34, №2, 1970.
2. Lur'e, K. A., Optimum control of the conductance of a fluid moving in a channel in a magnetic field. PMM Vol. 28, №2, 1964.
3. Lur'e, K. A., The Meyer-Bolza problem for multiple integrals and the optimization of performance of systems with distributed parameters. PMM Vol. 27, 1963
4. Mikhlin, S. G., Variational Methods in Mathematical Physics. Gostekhizdat, Moscow, 1957.
5. Vainberg, M. M. and Trenogin, V. A., The Theory of Solution Branching of Nonlinear Equations. "Nauka", Moscow, 1969.
6. Sutton, G. W., Design considerations of a magnetohydrodynamic electrical power generator. Vista Astronautica, Vol. 3, pp. 53-64, (N. Y. Soc. Automat. Engrs.), 1960.
7. Vatazhin, A. B. and Nemkova, N. G., Certain two-dimensional problems of electric current distribution in a magnetohydrodynamic channel with non-conducting baffles. PMTF, №2, 1964.

Translated by J. J. D.